

A new two-sphere singularity in general relativity

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Abstract

The Florides solution, proposed as an alternative to the interior Schwarzschild solution, represents a static and spherically symmetric geometry with vanishing radial stresses. It is regular at the center, and is matched to an exterior Schwarzschild solution. The specific case of a constant energy density has been interpreted as the field inside an Einstein cluster. In this work, we are interested in analyzing the geometry throughout the permitted range of the radial coordinate without matching it to the Schwarzschild exterior spacetime at some constant radius hypersurface. We find an interesting picture, namely, the solution represents a three-sphere, whose equatorial two-sphere is singular, in the sense that the curvature invariants and the tangential pressure diverge. As far as we know, such singularities have not been discussed before. In the presence of a large negative cosmological constant (anti-de Sitter) the singularity is removed.

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I. INTRODUCTION

The construction of theoretical models describing relativistic stars and the phenomenon of gravitational collapse is a fundamental issue in relativistic astrophysics. Pioneering work was done by Schwarzschild [1], who analyzed solutions describing a star of uniform energy density; Tolman provided explicit solutions of static fluid spheres [2]; Oppenheimer and Volkoff [3], by considering specific Tolman solutions, analyzed the gravitational equilibrium of stellar structures; Oppenheimer and Snyder [4], provided the first insights to gravitational collapse into a black hole; Buchdahl [5] and Bondi [6] also generalized the interior constant energy density solutions to more general static fluid spheres in the form of inequalities involving the energy density, central pressure and the location of the boundary matching surface. These authors, amongst others, lay down the foundations of the general relativistic theory of stellar structures (see Ref. [7] for an extensive review).

In the 1970's, Florides in an attempt to understand, within the framework of general relativity, why a spherically symmetric distribution of pressure-less dust at rest cannot maintain itself in equilibrium, discovered a new interior uniform density (Schwarzschild-like) solution. The latter solution is static, spherically symmetric, regular throughout the interior, and is matched to an exterior Schwarzschild spacetime [8]. It is interesting to note that the radial pressure is identically zero and the tangential pressure is positive and an increasing function of the radial coordinate. Now, at a first glance the absence of a radial pressure may cast doubts upon the physical significance of the Florides solution, as one is accustomed to thinking that it is precisely this radial pressure that maintains a system in static equilibrium. However, it was found to have a rather elegant physical interpretation, namely, the Florides interior solution describes the interior field of an Einstein cluster [9] (The Florides solution was further analyzed in Ref. [10]). Recall that the Einstein cluster describes a static and spherically symmetric gravitational field of a large number of particles moving in randomly oriented concentric circular orbits under their own gravitational field.

Whilst analyzing the Florides solution within itself, and considering the whole permitted range of the radial coordinate, without the respective matching to an exterior Schwarzschild spacetime at a junction interface, we came across an extremely interesting feature, namely, that the solution in fact represents a three-sphere, possessing a singular equatorial two-sphere, in the sense that the curvature invariants and the tangential pressure diverge. This

interesting aspect of the geometry motivated a more careful analysis of the Florides solution, as these two-sphere singularities have not been investigated before, to the best of our knowledge. However, it is interesting to note that in Ref. [11], by dropping the assumption of homogeneity on a cosmological scale, the authors considered a static and spherically symmetric model, containing a singularity which continually interacts with the Universe. It was suggested that the singularity can be interpreted as a sphere surrounding a regular central region. But, it is important to emphasize that the latter singularity is fundamentally different in nature to the two-sphere singularity analyzed in this work, as shall be discussed below in more detail.

We also stress that spacetime singularities have played a fundamental role in conceptual discussions of general relativity, and a key aspect of singularities in general relativity is whether they are a disaster for the theory, as they imply the breakdown of predictability. One may mention several attitudes that are widespread in the literature [12], namely, that singularities are mere artifacts of unrealistic and idealized models; general relativity entails singularities, but fails to accurately describe nature; and the existence of singularities may be viewed as a source to probe the limitations of general relativity, and from which one may derive a valuable understanding of cosmology [13]. We adopt the latter viewpoint throughout this work, attempting to understand the nature of the two-sphere singularity present in the Florides solution.

This paper is outlined in the following manner: In Section II, we deduce the general radial pressure-less solution, and further consider the specific case of constant energy density, with and without a cosmological constant, and provide the geometrical interpretation of the singular two-sphere. In Section III, we analyze specific characteristics of the geometry, such as the conserved quantities and geodesic motion. Finally, in Section IV, we conclude.

II. INTERIOR CONSTANT DENSITY SOLUTIONS RE-ANALYZED

A. General radial pressure-less solution

Consider a static and spherically symmetric spacetime, given in curvature coordinates, by the following line element

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2. \quad (1)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. As we are interested in analyzing solutions with a vanishing radial pressure, i.e., $p_r(r) = 0$, in the spirit of Ref. [8], the anisotropic stress energy tensor is given by

$$T_{\mu\nu} = \rho U_\mu U_\nu + p_\perp g_{\mu\nu}^\perp, \quad (2)$$

where $g_{\mu\nu}^\perp$ is the projection of the metric along the transverse spatial direction, i.e., orthogonal to the radial direction. It is defined as $g_{\mu\nu}^\perp = g_{\mu\nu} + U_\mu U_\nu - \chi_\mu \chi_\nu$, where U^μ is the four-velocity, and χ^μ is the unit spacelike vector in the radial direction, i.e., $\chi^\mu = e^{-\beta} \delta^\mu_r$. Note that $g_{\mu\nu}^\perp U^\nu = 0$, $g_{\mu\nu}^\perp \chi^\nu = 0$ and $U_\mu \chi^\mu = 0$. $\rho(r)$ is the energy density, and $p_\perp(r)$ is the transverse pressure measured in the orthogonal direction to χ^μ .

Using the Einstein field equation, $G_{\mu\nu} = 8\pi T_{\mu\nu}$ (with $c = G = 1$), the stress energy tensor components are given by

$$8\pi\rho(r) = \frac{e^{-2\beta}}{r^2} (2\beta'r + e^{2\beta} - 1), \quad (3)$$

$$8\pi p_r(r) = \frac{e^{-2\beta}}{r^2} (2\alpha'r - e^{2\beta} + 1) = 0, \quad (4)$$

$$8\pi p_\perp(r) = \frac{e^{-2\beta}}{r} [-\beta' + \alpha' + r\alpha'' + r(\alpha')^2 - r\alpha'\beta'] , \quad (5)$$

where the prime denotes a derivative with respect to the radial coordinate r .

Integration of Eq. (3) yields the following relationship

$$e^{-2\beta(r)} = 1 - \frac{2m(r)}{r}, \quad \text{with} \quad m(r) = 4\pi \int_0^r \rho(\bar{r}) \bar{r}^2 d\bar{r}, \quad (6)$$

where the integration constant has been evaluated by considering $\beta(0) = 0$. The function $m(r)$ is the quasi-local mass, and is denoted as the mass function. Substituting Eq. (6) in Eq. (4), we have

$$2\alpha(r) = \int_a^r \frac{2m(\bar{r})}{\bar{r}^2(1 - 2m(\bar{r})/\bar{r})} d\bar{r} + C, \quad (7)$$

where the constant of integration may be determined by matching this interior solution to a Schwarzschild exterior solution at a junction interface, a . Thus, the constant is given by $C = \ln(1 - 2M/a)$, where M is the object's mass, with $a > 2M$. With these relationships the tangential pressure is given by

$$p_\perp(r) = \frac{m(r)\rho(r)}{2r(1 - 2m(r)/r)}. \quad (8)$$

The latter relationship may also be obtained from the conservation of the stress energy tensor, $\nabla_\nu T^{\mu\nu} = 0$, which provides the anisotropic form of the Tolman-Oppenheimer-Volkoff

(TOV) equation. The metric finally assumes the form

$$ds^2 = - \left(1 - \frac{2M}{a}\right) \left[\exp \int_a^r \frac{2m(\bar{r}) d\bar{r}}{\bar{r}^2(1 - 2m(\bar{r})/\bar{r})} \right] dt^2 + \frac{dr^2}{1 - 2m(r)/r} + r^2 d\Omega^2. \quad (9)$$

B. Constant energy density

Considering a constant energy density [8], the mass function, given by Eq. (6), and Eq. (7) are readily integrated. This provides metric (9) in the following simplified form

$$ds^2 = - \frac{\left(1 - \frac{8\pi}{3}\rho_0 a^2\right)^{3/2}}{\left(1 - \frac{8\pi}{3}\rho_0 r^2\right)^{1/2}} dt^2 + \frac{dr^2}{1 - \frac{8\pi}{3}\rho_0 r^2} + r^2 d\Omega^2, \quad (10)$$

and corresponds to the following stress energy tensor

$$\rho(r) = \rho_0, \quad p_r(r) = 0, \quad p_\perp(r) = \frac{2\pi\rho_0^2 r^2}{3(1 - \frac{8\pi}{3}\rho_0 r^2)}, \quad (11)$$

where a is the matching surface, and the time was scaled to match with the Schwarzschild exterior. We immediately verify some problems with the above stress energy tensor components in comparison with the usual Schwarzschild interior solution. In the case of an isotropic constant density perfect fluid, the field equations yield a closed system of equations that can be solved uniquely for any given central pressure by means of the Tolman-Oppenheimer-Volkoff equation. The resulting pressure function turns out to be monotonically decreasing and its vanishing uniquely defines the boundary of the stellar object at which the Schwarzschild exterior metric can be matched.

Looking at the above tangential pressure component, given by Eq. (11), we realize that the pressure is monotonically increasing, and in fact diverges as $r \rightarrow 1/\sqrt{8\pi\rho_0/3}$. Hence, in contrast to the Schwarzschild interior solution, there is no ‘preferred’ vanishing pressure surface that is implied by the field equations. Thus one may match this interior solution at any value $2M < a < 1/\sqrt{8\pi\rho_0/3}$ to an exterior Schwarzschild spacetime and thereby avoiding the discussion of the singularity at $r = 1/\sqrt{8\pi\rho_0/3}$. This feature of the solution motivates the geometrical analysis of the global spacetime described by metric (10). Florides and later authors certainly noticed the divergent tangential pressure but presumably avoided its discussion by requiring that the Schwarzschild metric is matched at some smaller radius.

The chosen coordinate system for metric (10) is defined for radii that satisfy $r < R$. It is however easy to introduce a coordinate system that yields a much better geometrical

understanding. Therefore, let us introduce a third angle α defined by

$$r = R \sin \alpha, \quad \text{with} \quad 1/R = \sqrt{\frac{8\pi\rho_0}{3}}, \quad (12)$$

for which the metric (10) becomes

$$ds^2 = -\frac{\cos^3 \alpha_b}{\cos \alpha} dt^2 + R^2 d\alpha^2 + R^2 \sin^2 \alpha d\Omega^2, \quad (13)$$

where $a = R \sin \alpha_b$. The metric coefficient g_{tt} can be rescaled to have $g_{tt}(\alpha = 0) = 1$, which leads to the form of the metric that will be used henceforth

$$ds^2 = -\frac{dt^2}{\cos \alpha} + R^2 d\alpha^2 + R^2 \sin^2 \alpha d\Omega^2. \quad (14)$$

In performing a more careful analysis of the Florides solution we come across an extremely interesting aspect, namely, that the solution in fact represents a three-sphere, possessing a singular equatorial two-sphere. Before analyzing the geometry of the spacetime in more detail, we shall briefly consider the inclusion of a cosmological constant.

C. Presence of a cosmological constant

Chongming *et al* [14] extended the original work of Florides by taking into account the cosmological constant. This generalization yields the following metric and stress energy tensor, given by

$$ds^2 = -\frac{\left(1 - \frac{8\pi}{3}\rho_0 a^2 - \frac{\Lambda}{3}a^2\right)^{3/2(1+\Lambda/8\pi\rho_0)}}{\left(1 - \frac{8\pi}{3}\rho_0 r^2 - \frac{\Lambda}{3}r^2\right)^{(1-\Lambda/4\pi\rho_0)/2(1+\Lambda/8\pi\rho_0)}} dt^2 + \frac{dr^2}{1 - \frac{8\pi}{3}\rho_0 r^2 - \frac{\Lambda}{3}r^2} + r^2 d\Omega^2, \quad (15)$$

and

$$\rho(r) = \rho_0, \quad p_r(r) = 0, \quad p_\perp(r) = \frac{2\pi\rho_0^2 r^2}{3\left(1 - \frac{8\pi}{3}\rho_0 r^2 - \frac{\Lambda}{3}r^2\right)} \left(1 - \frac{\Lambda}{4\pi\rho_0}\right), \quad (16)$$

respectively. It is interesting to note that the spatial geometry now depends on the cosmological constant, see e.g. [15, 16, 17, 18, 19]. Let us introduce a new parameter k defined by

$$k = \frac{8\pi\rho_0}{3} + \frac{\Lambda}{3}. \quad (17)$$

For $k > 0$ the spatial geometry corresponds to a space of constant positive curvature, i.e., a sphere; for $k = 0$ the geometry is Euclidean; and for $k < 0$ the spatial geometry has constant negative curvature and is therefore hyperbolic. It is then also useful to introduce, similar to the third angle in the spherical case, adapted coordinates for the hyperbolic case. Hence, define

$$r = \frac{1}{\sqrt{|k|}} \sinh \alpha, \quad (18)$$

so that for the specific case of $k < 0$, metric (15) takes the form

$$ds_{k<0}^2 = -\frac{\cosh^3 \alpha_b^{1/(1+\Lambda/8\pi\rho_0)}}{\cosh \alpha^{(1-\Lambda/4\pi\rho_0)/(1+\Lambda/8\pi\rho_0)}} dt^2 + \frac{1}{|k|} (d\alpha^2 + \sinh^2 \alpha d\Omega^2). \quad (19)$$

Similarly, the stress energy tensor components simplify in the new coordinates to the following relationships

$$\rho(r) = \rho_0, \quad p_r(r) = 0, \quad p_\perp(r) = \frac{2\pi\rho_0^2}{|k|} \left(1 - \frac{\Lambda}{4\pi\rho_0}\right) \tanh^2 \alpha. \quad (20)$$

In contrast to the spherical case, the tangential pressure does not diverge in the hyperbolic case since $\lim_{\alpha \rightarrow \infty} \tanh \alpha = 1$. Furthermore, the center $\alpha = 0$ is regular (even flat) and therefore this hyperbolic spacetime is a globally regular spacetime, completely filled with an anisotropic perfect fluid, having constant energy density and vanishing radial pressure. Moreover, the Schwarzschild-anti de Sitter spacetime can be matched at any $\alpha = \text{constant}$ hypersurface so that the induced metric and the extrinsic curvature with respect to the matching surface are both continuous.

For the specific Euclidean case, where $k = 0$, one has to be careful by appropriately taking the limit in metric (15) because of the exponent. For consistency of the notation, let us rename r by α , so that the metric reads

$$ds_{k=0}^2 = -\exp\left(4\pi\rho_0(\alpha^2 - \alpha_b^2)\right) dt^2 + (d\alpha^2 + \alpha^2 d\Omega^2), \quad (21)$$

having the stress energy tensor components

$$\rho(\alpha) = \rho_0, \quad p_r(\alpha) = 0, \quad p_\perp(\alpha) = 2\pi\rho_0^2 \alpha^2. \quad (22)$$

It should also be noted that the stress energy tensor (16) implies that for $\Lambda = 4\pi\rho_0$, the tangential pressure also vanishes. In this case the stress energy tensor reduces to pressureless dust, $\rho(r) = \rho_0$. Since the k is also positive, the spatial geometry is spherical and hence

this spacetime is the original Einstein static universe (pressure-less) that was suggested by Einstein in 1917. However, the spherical case with non-vanishing tangential pressure does not allow the construction of an anisotropic Einstein static universe with vanishing radial pressure. It should be noted that the Einstein static universe can be generalized to have non-constant pressure with two regular centers, see [17, 18, 20, 21], and also in spacetimes with torsion an analog Einstein universe can be constructed, containing a constant radially symmetric torsion field [22]. It seems therefore that the anisotropic Einstein static universe is much more difficult to construct and it may possibly require a non-constant energy density. A similar question has not been answered yet (as far as we know), namely if a charged Einstein universe can in principle be constructed that also is globally regular.

Another interesting feature of an anisotropic matter distribution is their recent appearance in a rather different context of gravastars and dark energy stars (see e.g. Ref. [23] and references therein). For a constant energy density, the metric reported in [23] takes the form

$$ds^2 = -(1 - 2Ar^2)^{-(1+3w)/2} dt^2 + \frac{dr^2}{1 - 2Ar^2} + r^2 d\Omega^2, \quad (23)$$

where, as above, one can easily introduce a new coordinate (third angle) by $r = (1/\sqrt{2A}) \sin \alpha$. One should note that w , the dark energy equation of state parameter, and A can both be chosen so that this metric agrees with either the Florides metric (10), where simply $w = 0$, or its generalization due to the presence of Λ , Eq. (15). However, it is important to emphasize that to be a gravastar (or a dark energy star) solution a fundamental ingredient is a repulsive interior spacetime. This differs from the Florides solution, as in the latter the interior geometry is attractive.

D. Two-sphere singularity

We now turn to the analysis of the nature of the singularity in metric (14), for the spherically symmetric case with $\Lambda = 0$, although the main results that follow are unchanged for $k > 0$. For that, let us compute the non-vanishing Riemann tensor components

$$R_{\alpha t}{}^{\alpha t} = \frac{\cos^2 \alpha - 3}{4R^2 \cos^2 \alpha}, \quad (24)$$

$$R_{\alpha \theta}{}^{\alpha \theta} = R_{\alpha \phi}{}^{\alpha \phi} = R_{\theta \phi}{}^{\theta \phi} = \frac{1}{R^2}, \quad (25)$$

$$R_{\theta t}{}^{\theta t} = R_{\phi t}{}^{\phi t} = -\frac{1}{2R^2}, \quad (26)$$

and Weyl tensor components

$$C_{\alpha\theta}{}^{\alpha\theta} = C_{\alpha\phi}{}^{\alpha\phi} = C_{\theta t}{}^{\theta t} = C_{\phi t}{}^{\phi t} = \frac{\tan^2\alpha}{8R^2}, \quad (27)$$

$$C_{\alpha t}{}^{\alpha t} = C_{\theta\phi}{}^{\theta\phi} = -\frac{\tan^2\alpha}{4R^2}, \quad (28)$$

respectively. Since these Weyl tensor components are non-vanishing, we note crucial geometrical differences between the interior Schwarzschild solution and the Florides solution. It is well-known that the Schwarzschild interior solution is conformally flat [15, 24], irrespective of the cosmological constant [17, 18].

These yield the squared Riemann and Weyl tensors, and we also note the square of the Ricci tensor

$$\text{RiemSq} = \frac{3(68 \cos(2\alpha) + 19 \cos(4\alpha) + 73)}{32R^4 \cos^4\alpha}, \quad (29)$$

$$\text{WeylSq} = \frac{3 \tan^4\alpha}{4R^4}, \quad (30)$$

$$\text{RicciSq} = \frac{9(28 \cos(2\alpha) + 9(\cos(4\alpha) + 3))}{64R^4 \cos^4\alpha}, \quad (31)$$

respectively. All three geometrical invariants diverge near $\alpha = \pi/2$ in a similar way, by which we mean

$$\lim_{\alpha \rightarrow \pi/2} \frac{\text{WeylSq}}{\text{RicciSq}} = \frac{2}{3}, \quad (32)$$

$$\lim_{\alpha \rightarrow \pi/2} \frac{\text{RiemSq}}{\text{RicciSq}} = \frac{2}{3}, \quad (33)$$

$$\lim_{\alpha \rightarrow \pi/2} \frac{\text{WeylSq}}{\text{RiemSq}} = \frac{1}{2}, \quad (34)$$

namely, the Weyl tensor is not dominated by the Ricci tensor [25] and the singularity does not correspond to an isotropic singularity. The Ricci scalar is given by

$$g^{\mu\nu} R_{\mu\nu} = \frac{3(2 - 3 \sin^2\alpha)}{2R^2 \cos^2\alpha}, \quad (35)$$

which also diverges at $\alpha = \pi/2$.

It is interesting to note that such a singularity is not point-like. It describes a singular two-sphere, but the spacetime is well defined for $\alpha \in [0, \pi/2)$. Since the spatial part of this spacetime is a three-sphere we find the following geometrical picture: a three-sphere whose equatorial two-sphere is singular in the sense that the above invariants and the tangential

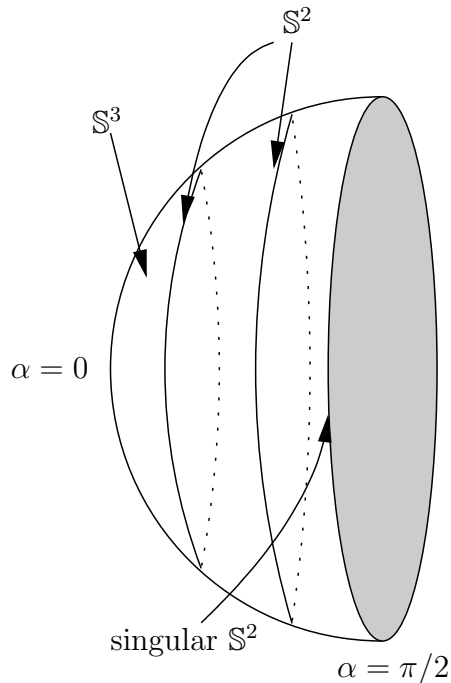


Figure 1: This figure represents the spatial three-sphere \mathbb{S}^3 . Vertical cuts through the three-sphere define the two-spheres \mathbb{S}^2 of the spherically symmetric spacetime. The equatorial cut through \mathbb{S}^3 at $\alpha = \pi/2$ defines the singular two-sphere \mathbb{S}^2 .

pressure diverge. However, the radial pressure (identically zero) and the energy density are both finite at the singularity. Fig. 1 represents the spatial three-sphere \mathbb{S}^3 .

It should be noted however, that the metric (14) is actually well defined for $\alpha = (-\pi/2, \pi/2)$. Therefore, we could in principle draw a second copy of the three-sphere so that the two half three-spheres are joined at $\alpha = 0$ rather than at the equator. However, we can still identify both singular two-spheres and would obtain something like Fig. 2.

The Florides solution can be interpreted as the interior of an Einstein cluster, therefore the singularity could also be interpreted from that point of view. We have a large number of particles that move in oriented circular orbits. Their individual velocities and angular momenta [26] are related to the tangential pressure and therefore the singular two-sphere corresponds to the surface where the particles all move with the speed of light. Obviously, such a surface has singular properties and it is expected that the proper time of the geodesics is zero, which is shown explicitly in the next Section. Furthermore our analysis seems to be important also in the context of rotating magnetized stellar objects. There exists a similar phenomena with respect to the rotating magnetic field lines. At large distances

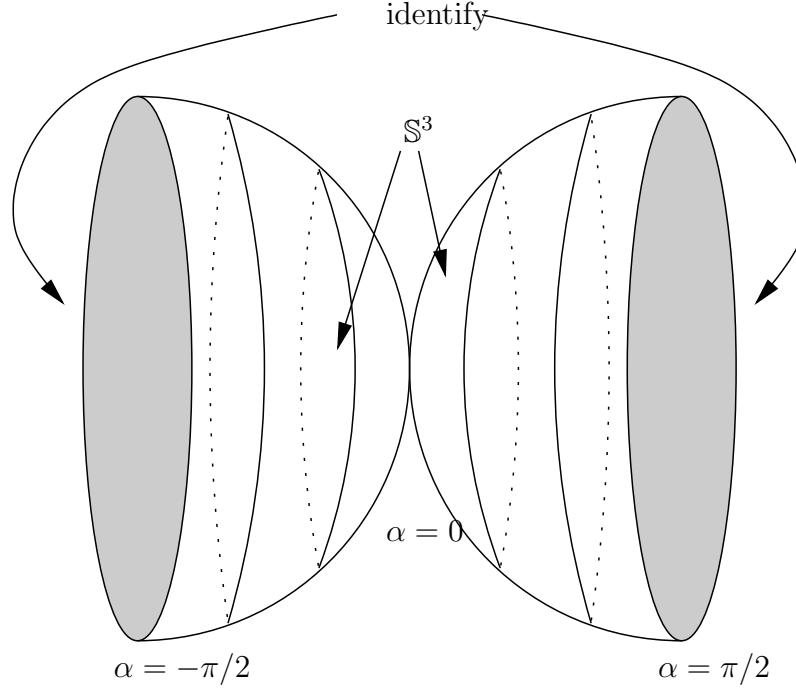


Figure 2: This figure represents the two half spatial three-spheres \mathbb{S}^3 . At $\alpha = 0$ they have a common point and we identify both singular two-spheres.

from the surface of the stellar object these line would rotate with the speed of light. A more detailed study comparing the geometry of the rotating magnetic field lines with the analytically simple Florides solution might shed some new light to the theory of magnetized stellar objects.

As referred to in the Introduction, in Ref. [11], the authors proposed a static and spherically symmetric model of the Universe, containing a singularity which can be viewed as a sphere surrounding a central region C , at $r = 0$. The solution possesses two centers, one at $r = 0$ and the second at $r = R$, as the surface area of the two-spheres of symmetry tends to zero at both centers. All past radial null geodesics intersect the singularity, as do all space-like radial geodesics. Thus, the singularity can be interpreted as surrounding the central region C , and lies within a finite distance from C , so that the Universe is spatially finite and bounded. Note that traversing a radial null geodesic from the singularity, one reaches the central region after a distance R , and the singularity is attained again after traveling a total distance of $2R$. The spacetime can be thought of as being spherically symmetric about both C and the singular center (we refer the reader to Ref. [11] for details). Note that the nature of this singularity is fundamentally different to the two-sphere singularity in

the Florides solution. First, in Ref. [11] the stress energy tensor components tend to zero as the singularity is attained at $r \rightarrow R$, while in the Florides solution the tangential pressure diverges. Second, the geometric structure is different, as the surface area of the Florides two-sphere is monotonically increasing, contrary to the case analyzed in Ref. [11], where the surface area is zero at $r = 0$ and at the singularity $r = R$. Thirdly, the cases analyzed in Ref. [11] impose $g_{tt} \rightarrow 0$ at $r \rightarrow R$ (although the case $g_{tt} \rightarrow \infty$ is briefly hinted at, it is not analyzed), while in the Florides solution we have $g_{tt} \rightarrow \infty$ as $r \rightarrow R$.

III. THE GEODESIC STRUCTURE OF THESE SOLUTIONS

A. Conserved quantities

Throughout this section, we shall consider metric (14). Consider the following Lagrangian

$$\mathcal{L}(x^\mu, \dot{x}^\mu) = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu. \quad (36)$$

If the metric tensor does not depend on a determined coordinate, x^μ , through the Euler-Lagrange equations one obtains that the quantity

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = g_{\mu\nu} \dot{x}^\nu, \quad (37)$$

is constant along any geodesic. Applied to line element (14), one verifies that the metric tensor components are independent of the coordinates t and ϕ , so that the conserved quantities are given by

$$\pi_\phi = g_{\phi\phi} \dot{\phi} = R^2 \sin^2 \alpha \dot{\phi} = L, \quad (38)$$

$$\pi_t = g_{tt} \dot{t} = -\frac{\dot{t}}{\cos \alpha} = -\mathcal{E}. \quad (39)$$

\mathcal{E} and L may be interpreted as the energy and angular momentum per unit mass. Without a loss of generality we may consider the equatorial plane with $\theta = \pi/2$.

The line element (14) may be rewritten in terms of the constants defined above, for the particular case of $\theta = \pi/2$, in the following manner

$$R^2 \dot{\alpha}^2 = \mathcal{E}^2 \cos \alpha + \left(2\mathcal{L} - \frac{L^2}{R^2 \sin^2 \alpha} \right), \quad (40)$$

where $\mathcal{L} = 0$ is defined for null geodesics, and $\mathcal{L} = -1/2$ for timelike geodesics.

The values of \mathcal{E} and L are determined by the initial conditions of the movement. For instance, consider a fixed observer along a point on the geodesic. The velocity of a test geodesic particle (see Ref. [27] for details), as measured by the observer is given by

$$V^2 = \frac{\cos \alpha}{\dot{t}^2} (R^2 \dot{\alpha}^2 + R^2 \sin^2 \alpha \dot{\phi}^2), \quad (41)$$

and substituting Eq. (38)-(40), we have

$$\mathcal{E}^2 = \frac{1}{(1 - V^2) \cos \alpha}, \quad (42)$$

for timelike geodesics, $\mathcal{L} = -1/2$. If a body initiates its movement at $\alpha = 0$ ($r = 0$) with $v = 0$, then $\mathcal{E} = 1$. Note that at $\alpha = \pi/2$ ($r = R$), we have $\mathcal{E} = \infty$. Indeed, the range of \mathcal{E} is precisely $1 < \mathcal{E}^2 < \infty$, as shall also be shown below.

B. Geodesics

1. Null geodesics

Consider null geodesics along the α -direction (r -direction), i.e., with $d\theta = d\phi = 0$, so that $dt = \pm \sqrt{\cos \alpha} R d\alpha$. Integrating the latter provides the following solution

$$t = \pm 2R E(\alpha/2, 2) + C_1, \quad (43)$$

where $E(\alpha, m)$ is the elliptic function of the second kind, defined as

$$E(\alpha, m) = \int_0^\alpha \sqrt{1 - m \sin^2 \varphi} d\varphi. \quad (44)$$

Consider the specific case of circular orbits, i.e., $\theta = \pi/2$ and $d\alpha = 0$, so that the line element reduces to

$$ds^2 = -\frac{dt^2}{\cos \alpha} + R^2 \sin^2 \alpha d\phi^2. \quad (45)$$

The null geodesic, $ds^2 = 0$, provides $dt/d\phi = \pm R \sin \alpha \sqrt{\cos \alpha}$, which has the following solution

$$t = 2\pi R \sin \alpha \sqrt{\cos \alpha}. \quad (46)$$

Note an interesting feature of this spacetime, namely, the time coordinate tends to zero as $\alpha \rightarrow \pi/2$ ($r \rightarrow R$), for null circular geodesics. Indeed, the time coordinate increases from $0 \leq \alpha < \arcsin \sqrt{2/3}$ ($0 < r < R\sqrt{2/3}$), and decreases from $\arcsin \sqrt{2/3} \leq \alpha < \pi/2$ ($R\sqrt{2/3} < r < R$). This is plotted in Fig. 3.

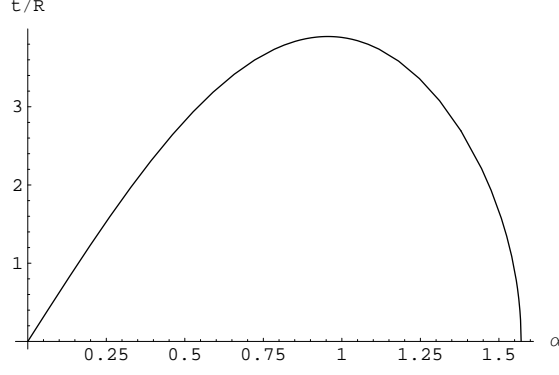


Figure 3: Consider the plot of circular null geodesics $t/R = 2\pi \sin \alpha \sqrt{\cos \alpha}$. Note that the time coordinate attains a maximum at $\alpha = \arcsin \sqrt{2/3}$, and tends to zero as $\alpha \rightarrow \pi/2$ ($r \rightarrow R$).

2. Timelike geodesics

For the case of an observer at rest with respect to the spacetime geometry, i.e., with (α, θ, ϕ) fixed, we verify that the relationship between the coordinate time and the proper time measured by the observer is given by

$$t = \pm \sqrt{\cos \alpha} \tau, \quad (47)$$

where the constant of integration has been defined as $t = 0$ for $\tau = 0$.

Consider the specific case of $\theta = \pi/2$ and $d\phi = 0$, which implies $L = 0$. Thus, we have to solve the following differential equation

$$\frac{d\tau}{d\alpha} = \pm R(\mathcal{E}^2 \cos \alpha - 1)^{-1/2}. \quad (48)$$

Note the restriction $\mathcal{E}^2 > 1/\cos \alpha$. Integrating the latter differential equation, we obtain

$$\tau(\alpha) = \frac{\pm R}{\sqrt{\mathcal{E}^2 - 1}} F(\alpha/2, 2\mathcal{E}^2/(\mathcal{E}^2 - 1)) + C_2, \quad (49)$$

where $F(\alpha, m)$ is the elliptic function of the first kind

$$F(\alpha, m) = \int_0^\alpha \frac{1}{\sqrt{1 - m \sin^2 \varphi}} d\varphi. \quad (50)$$

For the case of $\alpha = \text{const}$, we have

$$-1 = -\frac{\dot{t}^2}{\cos \alpha} + R^2 \sin^2 \alpha \dot{\phi}^2. \quad (51)$$

Using the relationship $L = R^2 \sin^2 \alpha \dot{\phi}$, the latter provides the following solution

$$t = \pm \sqrt{\cos \alpha \left(1 + \frac{L^2}{R^2 \sin^2 \alpha} \right)} \tau, \quad (52)$$

which reduces to Eq. (47) if $L = 0$. Note that one may also find a relationship for the proper time measured by an observer traversing a circumference at $\alpha = \text{const}$, in terms of \mathcal{E} , given by

$$\tau = \frac{2\pi R \sin \alpha}{\sqrt{\mathcal{E}^2 \cos \alpha - 1}}. \quad (53)$$

IV. SUMMARY AND DISCUSSION

In this work we took a fresh look at the Florides solution, which represents an interior static and spherically symmetric perfect fluid spacetime with vanishing radial stresses. In the standard approach to its physical interpretation, the Schwarzschild vacuum spacetime is matched at some constant radius hypersurface. However, we were interested in the complete geometry of the matter and therefore analyzed the geometry throughout the permitted range of the radial coordinate without requiring the matching to an exterior Schwarzschild spacetime. The resulting geometry is particularly interesting since it admits a two-sphere singularity which itself is the equator of a higher dimensional three-sphere. This is quite contrary to the usual scenario where the singularities are point-like.

The constant density Florides solution has an elegant interpretation as the field inside an Einstein cluster which is generated by particles moving in concentric circular orbits around the center. In view of this picture the singular two-sphere can be interpreted as the surface where all the particles are moving with the speed of light, and consequently our spacetime picture breaks down as particles ‘behind’ the singularity would move faster than the speed of light.

In conclusion, we emphasize that spacetime singularities have played a fundamental role in conceptual discussions of general relativity. A key aspect of singularities in general relativity is whether they are a disaster for the theory, as they imply the breakdown of predictability. In this work, we have adopted the attitude that the existence of singularities may be viewed as a source to probe the foundations and limitations of general relativity, and from which one may derive a valuable understanding of gravitation, and in this context analyzed a particularly interesting new type of a two-sphere singularity.

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